Probing Some Key Junctures in Relational Thinking: A Study of Year 6 and Year 7 Students from Australia and China

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This study uses number sentences involving two unknown numbers to identify some key junctures between relational thinking on number sentences and an ability to deal with sentences involving literal symbols. In this paper, the focus is on how students were able to make generalisations on sentences involving two unknown numbers, and how these influenced their performance on sentences involving literal symbols. In so doing, it aims to identify some key linkages as students make a transition from arithmetic to algebra.

Rationale for the Study

In its report *Algebra: Gateway to a Technological Future*, The Mathematical Association of America (2007) argues that "we need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and what are the critical *junctures* of this development" (p.2). This study, along with Stephens and Wang (2008), is about exploring some of those junctures, connections or linkages in relation to two kinds of mathematical sentences. The first type of sentence involves numbers and boxes either side of the equal sign, such as 18 + (Box A) = 20 + (Box B). A second type of sentence is structurally similar but uses unknowns, such as $c + 2 = d + 10$. Sentences of both types are used involving all four operations where the preceding two sentences exemplify addition.

Earlier research, such as carried out by Irwin and Britt (2005) focussed on Year 8 students' capacity to transform number sentences such as 47 + 25 into 50 + 22 by adding 3 to 47 and subtracting 3 from 25. Irwin and Britt examined the performances of nearly 900 12-year-old students in two schools which had participated in the New Zealand Numeracy Project and in two schools which had not. They argued that the methods of *compensating* and *equivalence* that some students use in transforming and solving number sentences, such as the one above, may provide a foundation for algebraic thinking (p. 169). They claimed that, when students apply strategies such as equivalence and compensation to sensibly solve different numerical problems, they disclose an understanding of the relationships of the numbers and operations involved; and "(t)hey show, without recourse to literal symbols, that the strategy is generalisable" (p. 171). Other authors, including Stephens (2007) and Carpenter and Franke (2001), refer to the thinking underpinning this kind of strategy as *relational* thinking. In all 21 questions used in the study by Irwin and Britt, students were required to demonstrate conceptual understanding of (relevant) relational strategies applied to six different sets of number sentences (p.174).

The *direction* of compensation that is appropriate for the operation of addition, for example, is inappropriate for subtraction (Kieran, 1981; Irwin & Britt, 2005). Some children who can successfully transform number sentences involving addition reason incorrectly, for example, that a number sentence such as 87 – 48 is equivalent to 90 – 45. Other children, however, use expressions such as, "in order for the difference to remain the

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same, the same number has to be *added* to each number in the expression to keep it equivalent (thus, 87 – 48 should be transformed to become 89 – 50)".

In order to examine students' capacity to use relational thinking across all four operations, and to investigate how they used key ideas of equivalence and compensation, studies by Stephens, Isoda, Inprasitha (2007) and Stephens (2007, 2008) had used number sentences involving four operations, where students were asked to find the value of a *single* missing number and to explain their thinking in sentences such as the following:

$43 + \square = 48 + 76$, $39 - 15 = 41 - \square$, $\square \times 5 = 20 \times 15$, $21 \div 56 = \square \div 8$

Some students demonstrated clearly relational strategies – either giving coherent verbal explanations showing how they had used equivalence and compensation or by using devices such as directed arrows – consistently across the range of sentences and operations. Others used computational methods clearly distinguishable from relational approaches. However, did they do so out of *choice* or was computation the *only* method they had at their disposal? All three above studies identified some students who *opted* for computational methods to deal with single-number missing number sentences, but *could* successfully apply ideas of equivalence and compensation to solve related sentences involving literal symbols. For these reasons, a clear computational strategy does not imply that such students are *incapable* of using a relational approach. Some may simply *prefer* to use a computational approach because it appears easier or is more familiar.

The current study therefore needed to include a different type of number sentence where all students are "pushed" to think relationally. Number sentences involving two unknown numbers, such as 18 + (Box A) = 20 + (Box B), seem to have this potential. These are called Type II number sentences to contrast with Type I number sentences which have a single missing or unknown number. While it is possible to use calculation to find particular instances of Type II number sentences like 18 + (Box A) = 20 + (Box B), identifying a general relationship requires students to move beyond computational thinking. Ideally, a student would say that the sentence will always be true *as long as the number in Box A is two more than the number in Box B*. But, in the classroom, how well and how clearly do students frame and express such generalisations? Being able to derive a correct mathematical generalisation from numerical examples is a key element of algebraic reasoning (Carpenter and Franke, 2001; Lee, 2001; Zazkis and Liljedahl, 2002).

Methodology

Design of Questionnaire

Type II number sentences – involving two unknown numbers – across all four operations – were used throughout the questionnaire. Initial questions might allow scope for computational approaches. But, following Fujii (2003), subsequent questions needed to push students towards identifying the relational elements embodied in these expressions, and to focus especially on expressing underlying mathematical generalisations.

Finally, to determine if equivalence and compensation, when used in Type II number sentences, provide a bridge to algebraic thinking, a question involving literal symbols was used for each operation in conjunction with its related Type II number sentence. These questions were modelled after the research programme, *Concepts in Secondary Mathematics and Science*, (CSMS, see Hart, 1981) which asked students, for example: What can you say about *c* and *d* in the following mathematical sentence?

$$c + 2 = d + 10$$

This type of question was called a Type III sentence. Structurally similar to Type II number sentences, it allowed students to say, ideally, that it will be true for *any* values of c and d provided c is 8 more than d. But some students may fall short of this explanation, simply giving specific values of c and d for which the sentence is true. Other students may say "c is more than d" but cannot fully specify the relationship. Such incomplete or partial interpretations may indicate different stages of development of relational thinking.

Type II and III: Sentences Involving Two Related but Unknown Numbers

This study focuses on four sets of questions, each representing one of the four operations, involving both Type II and Type III sentences. Each question was presented using a common template shown in Figure 1 which deals with the operation of addition. Type II questions are exemplified in parts (a) to (d) in Figure 1. These were then followed by a related Type III question, shown in part (e), involving the same arithmetical operation and using literal symbols.

Can you think about the following mathematical sentence:

18 + □ = 20 + □
Box A Box B

(a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

18 + □ = 20 + □
Box A Box B

18 + □ = 20 + □
Box A Box B

18 + □ = 20 + □
Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

(e) What can you say about c and d in this mathematical sentence? $c + 2 = d + 10$

Figure 1. Question involving addition and Type II and Type III questions.

Participants

The participants were drawn from Year 6 and Year 7 (12 to 14 years-old) students in two schools, one in Australia and one in China. The Chinese sample consisted of two intact classes consisting of 32 students in Year 6 and 36 students in Year 7. In the Australian school, one Year 6 class of 25 students was involved and three Year 7 classes consisting of 71 students altogether. The sample was a convenience sample. The performances of students are therefore not presented as being normative of schools in each country, and may reflect the teaching they have received. However, this limited sample allows one to examine students' performances on the two types of sentences, and to track what students do over certain junctures. Translation of the questionnaire into Chinese was prepared by faculty members at an Eastern Chinese university. Graduate students at the same university and two Chinese speaking graduates in Australia assisted with the translation of students' responses. Students' written responses were read independently by two markers. A very high degree of consistency of classification was evident across markers in both countries.

Key Questions to be Investigated in this Paper

The focus of this paper is about exploring some of those junctures, connections or linkages in relation to Type II and Type III sentences. Key issues to be investigated in this paper are how to classify the different kinds of responses to Type II and Type III sentences, and how performances on Type II sentences influenced performance on Type III sentences.

Results and Discussion

All students attempted the addition and subtraction questions involving Type II and III sentences. Some Year 6 students in the Chinese school had difficulty going any further, but this provided sufficient evidence. Year 6 students in the Australian school and Year 7 students in both schools generally completed all, or most of, the questionnaire.

Results on Type II and III Sentences

Type II and Type III sentences had been deliberately crafted to "push" students into relational responses especially those who may have completed parts (a) of these questions by computation. Almost all students without exception were able to place numbers correctly in Box A and Box B to make a correct sentence. Some students admittedly chose quite small numbers to place in the boxes to give correct sentences.

Having constructed several correct sentences in this way, all students attempted to describe the relationship between the numbers in Box A and Box B. However, there were clear differences in the way students described the relationship between the number in Box A and Box B or between *c* and *d*. These differences are shown in the following Table 1. Non-Directed Relational responses to part b and part c of Type II number sentences, as shown in Table 1, are incomplete but they are not wrong. Similar partially complete types of responses are exemplified below in Table 1 by Directed (no magnitude) Relational responses and Directed (non-referenced) Relational responses.

Fully Referenced and Directed Relational responses to part b of Type II sentences were evident in "B is 2 less than A" (Chinese Year 6 student answering part b for addition) and "Box A is 3 less than Box B" (Australian Year 7 student answering related part b question for subtraction)". Examples of fully relational responses to part d questions are "As long as B is 2 less than A" (Chinese Year 6 student answering the question involving addition) and the response by an Australian Year 7 student to question for part d involving subtraction who said "Any number can be in Box A, so long as Box A is 3 less than Box B". Students who used various *incomplete* expressions in part b questions to describe the relationship between Box A and Box B were *unable* to give a successful response to part d which asked "If you put any number in Box A, can you still make a correct sentence?"

Table 1
Describing the relationship between the numbers in Box A and Box B and between c and d

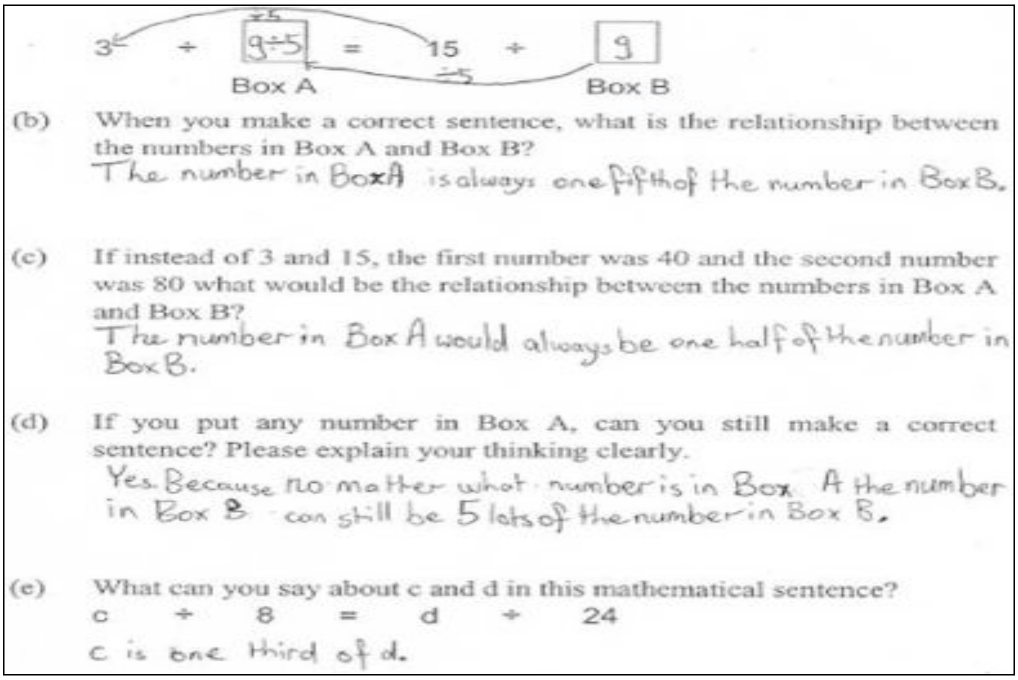
Response type	Examples
Incorrect Relation	Students continue to use ‘difference’ on multiplication and division question, as in – the difference (between c and d) is always 16 (as in $c \div 8 = d \div 24$)
Non-directed Relation	– they would always be *5 apart* [as in 3 ÷ (Box A) = 15 ÷ (Box B)] – there is always a *3 difference* [as in 72 – (Box A) = 75 – (Box B)] – the numbers have *a distance of 2* [as in 18 + (Box A) = 20 + (Box B)]
Directed (no magnitude) or Directed (non-referenced*) Relation	– so long as the number in Box B is larger [as in division or subtraction examples above] – d will be more than c [as in $c - 7 = d - 10$] – one number is always higher than the other number by 2 [as in addition example above] – one is 2 more than the other [as in addition example above]
Referenced Directed Relation	– one is 3 more than the other, Box B is bigger [as in subtraction example above] – c is 8 ahead of d [as in $c + 2 = d + 10$] – A is 5 times less than B [as in 3 ÷ (Box A) = 15 ÷ (Box B)] – difference of 2, A larger [as in 18 + (Box A) = 20 + (Box B)]

Note. “non-referenced” means not to point out the relational object, such as Box A and Box B, or *c* and *d*.

Categorisation of Types of Relational Thinking on Type II and III Questions

Based on the responses to Type II and III sentences across the four operations, we categorised relational thinking as Established Relational Thinking, Consolidating Relational Thinking, and Emerging Relational Thinking. In both schools in Year 6, many students still appeared to be operating as Emerging Relational Thinkers, less so in the Australian school as shown in Table 2 below. But in both schools by Year 7 the majority of students were able to show Consolidating or Established Relational Thinking. The defining characteristics of each of these three categories are as follows:

Established Relational thinkers successfully completed at least three of the four operations. As illustrated in Figure 2, they are able to: (a) specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the numbers, including the magnitude and direction of the difference between them; (b) employ a similar form of words used to describe this relationship as a part of the condition that describes how any number can be used in Box A and still make a true sentence; (c) explain clearly how c and d are related for the Type III sentence to be true, treating c and d as general numbers.



3 ÷ 9÷5 = 15 ÷ 9

Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

The number in BoxA is always one fifth of the number in BoxB.

(c) If instead of 3 and 15, the first number was 40 and the second number was 80 what would be the relationship between the numbers in Box A and Box B?

The number in Box A would always be one half of the number in Box B.

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

Yes. Because no matter what number is in Box A the number in Box B can still be 5 lots of the number in Box B.

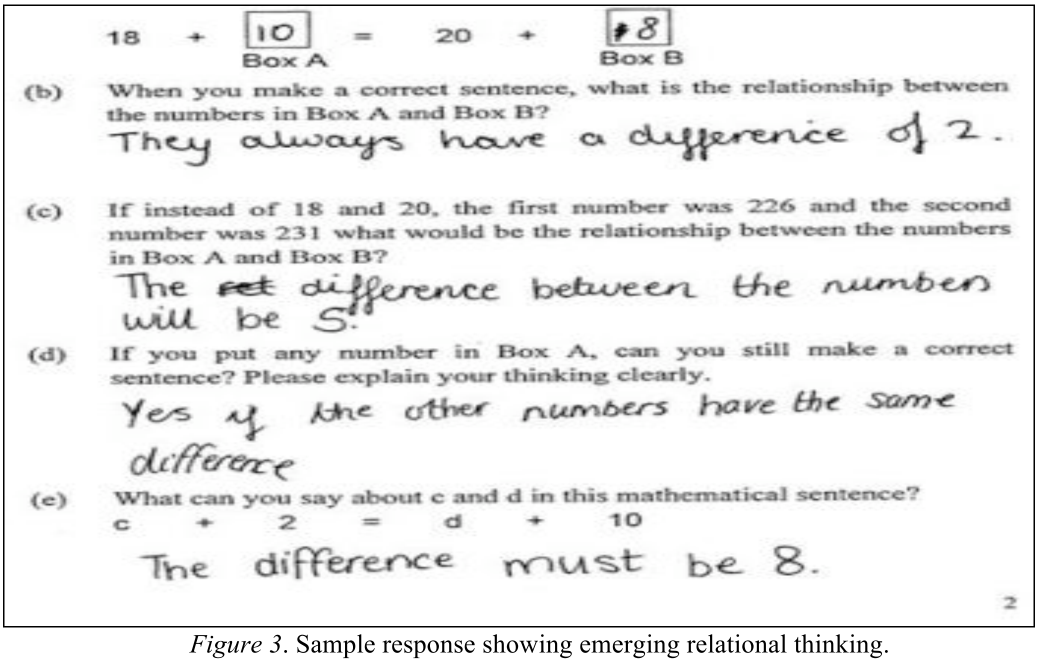
(e) What can you say about c and d in this mathematical sentence?

c ÷ 8 = d ÷ 24

c is one third of d.

Figure 2. Sample response showing established relational thinking.

Consolidating Relational thinkers, by contrast, almost always gave a full and correct specification in parts b and c of the relationships between the numbers in Box A and the numbers in Box B according to the operation under consideration; but were not able to give a consistent and correct response in part d to how any number might be used in Box A and still have a true sentence; and were only rarely able to give a complete explanation of the relationship between *c* and *d*, typically referring to some feature of the relationship (e.g. Directed no magnitude), or giving a specific pair of values for *c* and *d*. Those students who only completed questions relating to two of the four operations were also classified as Consolidating Relational Thinkers, even if their responses were fully correct.



18 + 10 = 20 + 8

Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

They always have a difference of 2.

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

The ~~set~~ difference between the number will be 5!

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

Yes if the other numbers have the same difference

(e) What can you say about c and d in this mathematical sentence?

c + 2 = d + 10

The difference must be 8.

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Figure 3. Sample response showing emerging relational thinking.

Emerging Relational thinkers– as illustrated in Figure 3 – typically: (a) identify some feature of the numbers used in Box A and Box B, but cannot completely specify the relationship between the numbers used in Box A and Box B; (b) focus on this feature when trying to explain how any number can be used in Box A and still have a true sentence, but are unable to describe this relationship completely; (c) may attempt to give a correct pair of values for which *c* and *d* might make a true sentence; or they focus on one aspect of this relationship, as in the response shown in Figure 3; or omit this question altogether.

Among Established and Consolidating Relational thinkers, shown in Table 2, there was a striking association between making a clear and correct response to part d and describing the relationship between the values of *c* and *d* to make the corresponding Type III sentence true. Among the 26 Year 6 students in both schools, classified as Consolidating or Established Relational thinkers, 70% of those who gave a correct response to a particular part (d) item also correctly described the relationship between *c* and *d* in the corresponding part (e) item. This same feature occurred in 90% of cases among the corresponding 89 Year 7 students, in both schools. In addition, a successful response to a part e item was rarely preceded by an inadequate response to its related part d.

Table 2

Student performances on Type II and III between Chinese and Australian students

School	Year	Emerging Relational	Consolidating or Established Relational
Chinese	Year 6	19	13
	Year 7	5	31[a]
Australian	Year 6	12	13[b]
	Year 7	13	58[c]

Notes: (a) 6 out these 31 students successfully completed all questions; (b) 3 out these 13 students successfully completed all questions; (c) 28 out these 58 students successfully completed all questions.

Conclusions and Implications

Analysing the performances of the four different groups allows several important conclusions to be drawn concerning some key junctures in relational thinking. Different student responses to parts b and c of Type II questions have different potential for completing part d relating to Type II and part e relating to Type III questions. If students did not completely specify the numbers in Box A and Box B and the relationship between them in parts b and c, they were without exception *unable* to specify a condition in part d that describes how any number might be used in Box A and still have a true sentence. Thus, Emerging Relational thinkers who answer part b (or part c) incompletely, appear unable to rectify this incompleteness in part d when asked to specify a condition that describes how any number might be used in Box A and still have a true sentence.

By contrast, a successful generalisation about Type II number sentences in part d was almost always followed by a successful explanation of the relationship between *c* and *d* in Type III sentences. Some Consolidating Relational thinkers who successfully completed a part e question did not successfully complete the related part d. These students may have treated part e as a kind of textbook question, without seeing any structural similarity between part d and part e. But the very close association between successfully completed parts d and parts e suggests that students saw and were able to capitalise on the structural similarity between the two questions. Established Relational thinkers either described a condition for the sentence involving *c* and *d* to be true, as in 'As long as *c* is 8 more than *d* ', or they wrote symbolic expressions for this condition such as $c - d = 8$ or $c = d + 8$.

A striking fact is that students who did not completely specify the numbers in Box A and Box B and the relationship between them in parts b and c for Type II sentences *never* answered part d successfully, nor could they specify a condition that describes the relationship between *c* and *d* in Type III sentences (part e). Emergent Relational Thinking characterised by responses to parts b and c that are either Non-directed, or Directed (no magnitude), or Directed (non-referenced) seems to lock students into forms of thinking that prevents them from making the kind of generalisations required for part d and e.

While these descriptions can be seen positively as denoting an early stage of relational thinking development, teachers need to help students to articulate fully referenced and directed relational descriptions. This may be done through highlighting to students the disadvantages and advantages that different descriptions offer.

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